Transient Effects in Solid Propellant Burning

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From theoretical considerations, stability criteria for solid propellant burning behavior are derived. The analysis considers the transient temperature field of the solid subjected to a cosine pressure perturbation in the gas stream. The results indicate that the modes of propellant burning behavior are influenced by the variation of the temperature field within the solid propellant. This effect is represented by the term $C = [v^2t_0/\alpha]^{1/2}$. The theory demonstrates that for different values of C, an initially stable propellant will burn in an unstable manner. In addition, the theory indicates a range in propellant parameters for which unstable burning always occurs.

Nomenclature

a	= summation index
A	= $E_w[T_w - T_i]/RT_w^2$, dimensionless parameter
b	= dimensionless parameter, see Eq. (25)
$2vB/\alpha$	= A, dimensionless parameter
C	= $[v^2t_0/\alpha]^{1/2}$, dimensionless initiation time parameter
c_p	= specific heat of gas
c_s	= specific heat of solid
D_1, D_2	= const
E_f	= activation energy for combustion
E_w	= activation energy for vaporization and decomposition
$E(t_0)$	= dimensionless function, see Eq. (25)
n	= order of gas-phase reaction
p'	= pressure of gas phase
p	= dimensionless parameter, see Eq. (9)
$P(\epsilon, t)$	= dimensionless function, see Appendix D
q	= dimensionless parameter, see Eq. (9)
$Q(\epsilon, t)$	= dimensionless function, see Appendix D
R	= universal gas constant
$R(\epsilon, t)$	= dimensionless function, see Appendix D
r	= summation index
r_1, r_2	= dimensionless modulus, see Eq. (30)
s	= LaPlace transform variable
$T(x, t)$	= unperturbed solid temperature
$\bar{T}(x, t)$	= total perturbed solid temperature
t	= reduced time
t'	= time
u	= $1 + (4s\alpha/v^2)$
u_i	= dimensionless roots
v	= unperturbed velocity of propellant surface
$\bar{v}$	= total perturbed velocity at propellant surface
x	= space
z	= complex variable

Subscripts

f	= flame
i	= initial
0	= reference
w	= surface

Special symbols

α	= thermal diffusivity of solid
ϵ	= vx/α, dimensionless parameter
θ	= LaPlace transform of perturbed temperature, $\int_0^\infty \delta T' e^{-st} dt$
ψ	= dimensionless parameter, see Eq. (12)
η	= dimensionless variable
ν	= summation index
$U(\eta)$	= step function
δ	= perturbed term
ω	= frequency of oscillation
φ_1, φ_2	= argument of a complex number
$\left.\dfrac{\delta p'}{p'}\right\|$	= amplitude of pressure perturbation
$\delta T'$	= $\dfrac{1}{2\pi i}\displaystyle\int_{\gamma - i\infty}^{\gamma + i\infty} \theta(\epsilon, s)e^{st}\, ds$
$\delta T'/nb\,\|\delta p'/p'\|$	= amplitude of temperature perturbation
arg$z$	= argument of a complex number
$R_e u_i$	= real part of the complex number u_i

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Introduction

To understand fully the characteristics of solid propellant burning, it would be of great value if the dynamic responses of the gas and solid phases, simultaneously, were known. It is apparent that there is an energy interchange between the gas stream and the solid propellant; hence any variations in one of the energy reservoir's behavior may produce a significant change in the other upon energy redirection. Under these circumstances, the effect that appears is an unwarranted change in pressure, temperature with time, or an altered mass release from the propellant which induces the forementioned effects. As shown in Refs. 1–3, studies of the gas stream behavior are possible when the presence of the solid propellant is approximated with suitable boundary conditions; the temperature history of the propellant is not accounted for in these studies.

Because the thermal response of the solid propellant influences the energy feedback to the gas stream, a detailed study of the solid propellant is required as well. Reference 4 and the Denison and Baum's study, Ref. 5, investigated a solid propellant burning in a varying pressure atmosphere. The gas stream behavior is connected to the propellant surface temperature via a simplified Arrhenius relationship. From small perturbation techniques, the analysis shows that the propellant can burn unstably for a range in physical parameters. This work assumes a priori that the preperturbed temperature field within the propellant changes with distance and not with time; hence the propellant is considered to be at its steady-state condition during perturbation.

Since time-dependent pressure variations in the gas stream produce temperature changes within the solid that are both time- and space-dependent, it would appear that the restriction of a steady-state temperature profile within the solid propellant is too severe. As a first approximation to the preceding condition, the present investigation extends the steady-state study of Ref. 5 to the transient case, i.e., a solid propellant whose preperturbed internal temperature is a function of time and distance. The results indicate that the stability criteria developed in Ref. 5 are strongly influenced by the initiation time of the pressure oscillation.

Analysis

In the analysis that follows, the solid propellant is considered to be semi-infinite with constant physical properties. The surface of the propellant is assumed to move with a constant surface temperature and velocity T_w and $\bar{v}$, respectively. Consequently, the one-dimensional equation for the temperature profile in the solid is

$$(\partial \bar{T}/\partial t') = \alpha(\partial^2 \bar{T}/\partial x^2) + \bar{v}(\partial \bar{T}/\partial x) \tag{1}$$

where α is the propellant thermal diffusivity, and T, x, and t' are the temperature, space, and time variables, respectively. In Eq. (1), the distance x is measured from the propellant surface, which is always located at the origin of the x axis. From small perturbation methods, the pressure, temperature, and velocity terms are approximated by an infinitesimal change in the preceding quantities. Hence

$$\begin{aligned} \bar{v} &= v + \delta v \\ \bar{T}(x, t') &= T(x, t') + \delta T(x, t') \\ \bar{p}' &= p' + \delta p' \end{aligned} \tag{2}$$

where v, p', and $T(x, t')$ represent the unperturbed or noneffected terms. Furthermore, it is assumed that the propellant surface obeys the simplified Arrhenius relation developed in Ref. 5. Thus the ratio of the perturbed velocity to the unperturbed value may be written as

$$\frac{\delta v}{v} = \frac{E_w}{RT_w}\left[\frac{\delta T(0, t')}{T_w}\right] \tag{3}$$

E_w is the activation energy associated with vaporization and decomposition of the solid propellant. Equation (1) may now be amended by the substitution of the perturbation quantities, Eq. (2), and the Arrhenius relation, Eq. (3); hence

$$\frac{\partial \delta T}{\partial t'} = \alpha \frac{\partial^2 \delta T}{\partial x^2} + v\frac{\partial \delta T}{\partial x} + \frac{vE_w}{RT_w}\left[\frac{\delta T(0, t')}{T_w}\right]\frac{\partial T}{\partial x} \tag{4}$$

From Ref. 12, the derivative of the unperturbed temperature profile for a semi-infinite solid with a constant face temperature is

$$\frac{\partial T}{\partial x} = -\left\{\frac{T_w - T_i}{2}\right\}\left\{\frac{v}{\alpha}\, e^{-vx/\alpha}\,\mathrm{erfc}\left[\frac{x - vt'}{2(\alpha t')^{1/2}}\right] + \frac{1}{(\pi\alpha t')^{1/2}}\left(\exp\left[-\frac{vx}{\alpha} - \frac{(x - vt')^2}{4\alpha t'}\right] + \exp\left[-\frac{(x + vt')^2}{4\alpha t'}\right]\right)\right\} \tag{5}$$

Substituting Eq. (5) and the following dimensionless quantities

$$\begin{aligned} \delta T' &= \delta T/(T_w - T_i) \\ \epsilon &= vx/\alpha \\ A &= 2vB/\alpha = \{E_w[T_w - T_i]\}/RT_w^2 \end{aligned} \tag{6}$$

into Eq. (4), the perturbation equation reduces to

$$\frac{\alpha}{v^2}\frac{\partial \delta T'}{\partial t'} = \frac{\partial^2 \delta T'}{\partial \epsilon^2} + \frac{\partial \delta T'}{\partial \epsilon} - B\delta T'(0, t')\left\{\frac{2}{(\alpha\pi t')^{1/2}}\exp\left[-\frac{\epsilon}{2} - \frac{\alpha\epsilon^2}{4v^2t'} - \frac{v^2t'}{4\alpha}\right] + \frac{v}{\alpha}\, e^{-\epsilon}\,\mathrm{erfc}\left[\frac{(\alpha\epsilon/v) - vt'}{2(\alpha t')^{1/2}}\right]\right\} \tag{7}$$

A solution to Eq. (7) is feasible if it is further assumed that the transient field in the solid propellant is not affected by the temperature perturbation $\delta T'$. This implies that, if a perturbation is introduced at some time $t' = t_0$, then the term appearing within the braces in Eq. (7) is constant for $t' \geq t_0$. This assumption is similar to the one made in the steady-state investigation; however the latter method does not incorporate any dependency upon t_0. Consequently, Eq. (7) is now amended by the replacement of the time term t' by a reduced time term $t = t' - t_0$ and the term t' in the brackets by the initiation time t_0. The form of the resultant differential equation is similar to Eq. (7); hence to avoid repetition the amended equation is not shown. Applying LaPlace transform techniques, the amended equation can be transformed into the ordinary differential equation

$$\frac{\alpha}{v^2}\, s\theta(\epsilon, s) = \frac{d^2\theta}{d\epsilon^2} + \frac{d\theta}{d\epsilon} - B\theta(0, s)\left\{\frac{2}{(\alpha\pi t_0)^{1/2}}\exp\left[-\frac{\epsilon}{2} - \frac{\alpha\epsilon^2}{4v^2t_0} - \frac{v^2t_0}{4\alpha}\right] + \frac{v}{\alpha}\, e^{-\epsilon}\,\mathrm{erfc}\left[\frac{(\alpha\epsilon/v) - vt_0}{2(\alpha t_0)^{1/2}}\right]\right\} \tag{8}$$

where $\theta(\epsilon, s)$ is the transform of the temperature. From the method of the variation of constants, the general solution to Eq. (8) is

$$\theta(\epsilon, s) = D_1e^{p\epsilon/2} + D_2e^{q\epsilon/2} + \frac{2vB}{\alpha}\,\theta(0, s)e^{st_0}\left\{\frac{e^{p\epsilon/2}\,\mathrm{erf}(1/2C)\,(\epsilon + C^2[1 + (4s\alpha/v^2)]^{1/2})}{-q} + \frac{e^{q\epsilon/2}\,\mathrm{erfc}(-1/2C)(\epsilon - C^2[1 + (4s\alpha/v^2)]^{1/2})}{-p} - \frac{v^2}{2\alpha s}\, e^{-\epsilon - st_0}\,\mathrm{erfc}\frac{1}{2C}\,[\epsilon - C^2]\right\}$$

where

$$\begin{aligned} p &= -1 + [1 + (4s\alpha/v^2)]^{1/2} \\ q &= -1 - [1 + (4s\alpha/v^2)]^{1/2} \end{aligned} \tag{9}$$

and $C = [v^2t_0/\alpha]^{1/2}$. The terms D_1, D_2, and $\theta(0, s)$ are eliminated from Eq. (9) by application of the boundary conditions

$$\begin{aligned} \lim_{\epsilon \to \infty} \theta(\epsilon, s) &\to 0 \\ \lim_{\epsilon \to 0} \theta(\epsilon, s) &\to \theta(0, s) \end{aligned} \tag{10}$$

and

$$\left.\frac{-d\theta}{d\epsilon}\right]_{\epsilon=0} = A\Psi\theta(0, s) + \frac{nc_pT_f}{c_s[n + 2 + (E_f/RT_f)][T_w - T_i]}\int_0^\infty \frac{\delta p'}{p'}\, e^{-st}\, dt \tag{11}$$

The boundary conditions represented by Eq. (10) follow from the mathematical model of semi-infinite solid with a constant face temperature. Equation (11) is an energy balance for the gas species liberated from the solid propellant and the heat flux to the wall. Since a similar equation is constructed in detail in Ref. 5, its derivation will not be repeated at this time. The terms A and Ψ appearing in Eq. (11) are defined to be

$$A = E_w[T_w - T_i]/RT_w^2$$

and

$$\Psi = \frac{1}{A} - \frac{2c_pT_f}{c_s[n + 2 + (E_f/RT_f)][T_w - T_i]} + E(t_0) \tag{12}$$

For the theoretical development that follows, the gas-pressure variation will be assumed as

$$\delta p'/p' = |\delta p'/p'|\cos\omega t \tag{13}$$

Applying Eqs. (10) and (11) into (9), the expression for the transformed temperature becomes

$$\theta(\epsilon, s) = \frac{\left\{\dfrac{-nc_pT_f|\delta p'/p'|v^2/4\alpha}{Bc_s[n + 2 + (E_f/RT_f)][T_w - T_i]}\right\}\left\{\dfrac{1}{s^2 + \omega^2}\right\}\left\{\dfrac{2vB}{\alpha} e^{st_0}\left[pe^{p\epsilon/2}\operatorname{erfc}\dfrac{1}{2C}\left(+ C^2\left(1 + \dfrac{4s\alpha}{v^2}\right)^{1/2}\right) + S(\epsilon) + V(\epsilon)\right]\right\}}{\dfrac{v}{\alpha} e^{st_0}\left[1 - \dfrac{p}{q}\right]\operatorname{erfc}\dfrac{C}{2}\left(1 + \dfrac{4s\alpha}{v^2}\right)^{1/2} - \dfrac{q}{2B} - \dfrac{v^3}{\alpha^2 s}\left[1 + \dfrac{q}{2}\right]\left[1 + \operatorname{erf}\dfrac{C}{2}\right] - \dfrac{2v}{\alpha}\Psi} \tag{14}$$

where

$$S(\epsilon) = -qe^{q\epsilon/2}\operatorname{erfc}\frac{-1}{2C}\left(\epsilon - C^2\left(1 + \frac{4s\alpha}{v^2}\right)^{1/2}\right) + 2e^{-st_0-\epsilon}\operatorname{erfc}\left(\frac{\epsilon}{2C} - \frac{C}{2}\right) \tag{14a}$$

and

$$V(\epsilon) = e^{q\epsilon/2}\left[-\frac{4s\alpha}{v^2} - \frac{2vB}{\alpha}e^{st_0}\left([p - q]\operatorname{erfc}\frac{C}{2}\left(1 + \frac{4s\alpha}{v^2}\right)^{1/2} + 2e^{-st_0}\operatorname{erfc}\frac{-C}{2}\right)\right] \tag{14b}$$

Hence, the temperature perturbation $\delta T'$ is obtained from the inverse theorem of LaPlace transforms. As shown in Ref. 5, the evaluation of the inversion integral can be expedited by a change in variables from s to u by the substitution $u = 1 + 4s\alpha/v^2$ in the resultant expression for the integral. Subsequently, $\delta T'(x, t)$ may be expessed as

$$\delta T'(x, t) = \left\{\frac{-nc_pT_f|\delta p'/p'|(v^2/4\alpha)^2}{c_s[n + 2 + (E_f/RT_f)][T_w - T_i]}\right\} e^{-v^2t/4\alpha} G\left(x, \frac{tv^2}{4\alpha}\right) \tag{15}$$

where, from Ref. 6,

$$G\left(x, \frac{tv^2}{4\alpha}\right) = \left(\frac{4\alpha}{\pi tv^2}\right)^{1/2}\int_0^\infty e^{-\alpha\eta^2/tv^2} L_\eta^{-1}\{ug(u^2)\}d\eta \tag{16}$$

From Eq. (14) it follows that

$$ug(u^2) = \sum_i \left\{\frac{1}{\prod\limits_{j=1}^{r}(u_j - u_i)}\right\}\left\{\frac{2vB}{\alpha} e^{(C^2/4)(u_i^2-1)}\left[u_i(u_i - 1)e^{(\epsilon/2)(u_i-1)}\operatorname{erfc}\frac{1}{2C}(\epsilon + u_iC^2) + u_i(u_i + 1)e^{-(\epsilon/2)(u_i+1)} \times \right.\right.$$
$$\left.\operatorname{erfc}\frac{-1}{2C}(\epsilon - u_iC^2)\right] + 2u_i\exp\left[\left(-\epsilon - \frac{C^2}{4}(u_i^2 - 1)\right)\operatorname{erfc}\frac{1}{2C}(\epsilon - C^2)\right] + e^{-(\epsilon/2)(u_i+1)}\left[u_i(1 - u_i^2) - \right.$$
$$\left.\left.\frac{4vB}{\alpha}u_i e^{(C^2/4)(u_i^2-1)}\left(u_i\operatorname{erfc}\left(\frac{u_iC}{2}\right) + e^{-(C^2/4)(u_i^2-1)}\operatorname{erfc}\left(\frac{-C}{2}\right)\right)\right]\right\} \tag{17}$$

Here u_j represents the rth general root, and $j \neq i$.

Since the denominator in Eq. (14) is a combination of transcendental functions of a complex number, the solution for the roots is generally intractable. However, a solution is possible if it is noted that propellants are considered to be near their steady-state condition; consequently the quantity $u_iC/2$ is assumed to be large. Thus for large values of this quantity, the denominator appearing in Eq. (16) may be approximated by

$$(u - u_1)(u - u_2) = \left[u - A\Psi + 1 + \left\{(A\Psi)^2 - \frac{4Ae^{-C^2/4}}{C(\pi)^{1/2}} - 2A\left(1 + \operatorname{erf}\left(\frac{C}{2}\right)\right)\right\}^{1/2}\right] \times$$
$$\left[u - A\Psi + 1 - \left\{(A\Psi)^2 - \frac{4Ae^{-C^2/4}}{C(\pi)^{1/2}} - 2A\left(1 + \operatorname{erf}\left(\frac{C}{2}\right)\right)\right\}^{1/2}\right] \tag{18}$$

The derivation for Eq. (18) is shown in Appendix A. There are four more roots associated with $L_\eta^{-1}ug(u^2)$ and these are evaluated in a straightforward manner from the transform of the pressure expression

$$L\frac{\delta p'}{p'} = \left|\frac{\delta p'}{p'}\right|\left\{\frac{s}{s^2 + \omega^2}\right\} = \left|\frac{\delta p'}{p'}\right|\left\{\frac{(v^2/4\alpha)[u^2 - 1]}{(v^2/4\alpha)^2[u^2 - 1]^2 + \omega^2}\right\} \tag{19}$$

Thus, the additional roots are

$$u_{3,4} = \pm [1 + i(4\alpha\omega/v^2)]^{1/2} \tag{20}$$
$$u_{5,6} = \pm [1 - i4\alpha\omega/v^2]^{1/2}$$

In order to obtain an explicit expression for the perturbed temperature $\delta T'$, Eq. (16) must be integrated in the manner shown. From the shifting theorems and the LaPlace transform pairs shown in Ref. 6, the inverse of Eq. (17), $L_\eta^{-1}ug(u^2)$, is readily obtainable. This expression is shown in Appendix B. In addition, some of the integrations with respect to the variable η are rather involved; hence, the reader is referred to Appendix C. Furthermore, due to the assumption that the quantity $(u_iC/2)$ is large, the temperature relationship now represents its asymptotic expression.[7] It can easily be shown that in the limit the resultant temperature expression, Eq. (21), becomes the steady-state temperature equation in Ref. 5. Upon integration, the asymptotic perturbed temperature is found to be

$$\frac{\delta T'(x, t)}{\dfrac{2vB/\alpha[-2nc_pT_f|\delta p'/p'|]}{c_s[n + 2 + (E_f/RT_f)][T_w - T_i]}} = 2\sum_{i=1}^{6}\frac{u_i(u_i + 1)}{\prod\limits_{j=1}^{6}(u_j - u_i)}\exp\left[-\epsilon - (1 - u_i^2)\frac{tv^2}{4\alpha}\right]\operatorname{erfc} - u_i\left(\frac{tv^2}{4\alpha}\right)^{1/2} \times$$
$$\operatorname{erfc}\frac{1}{2C}(\epsilon - C^2) + \sum_{i=1}^{6}\frac{u_i(u_i^2 - 1)}{\prod\limits_{j=1}^{6}(u_j - u_i)}\exp\left[-\frac{v^2}{4\alpha}(t + t_0)(1 - u_i^2) - \frac{\epsilon}{2}(1 - u_i)\right] \times$$

[*Equation (21) is continued on the next page.*]

$$
\begin{aligned}
&\Bigl\{\operatorname{erfc}- u_i\left(\frac{tv^2}{4\alpha}\right)^{1/2} \operatorname{erfc}\frac{1}{2C}(\epsilon+u_iC^2) - \left(\frac{4\alpha}{\pi t v^2}\right)^{1/2}\Bigl[\frac{C}{(\pi)^{1/2}}\exp\left(-\frac{1}{4C^2}(\epsilon+u_iC^2)^2\right) - \\
&\frac{1}{2}(\epsilon+u_iC^2)\operatorname{erfc}\frac{1}{2C}(\epsilon+u_iC^2)+P(\epsilon,t)\Bigr]\Bigr\} + \sum_{i=1}^{6} \frac{u_i(u_i+1)^2}{\prod\limits_{j=1}^{6}(u_j-u_i)}\exp\left[-\frac{v^2}{4\alpha}(t+t_0)(1-u_i^2)-\frac{\epsilon}{2}(1+u_i)\right]\times \\
&\Bigl\{\operatorname{erfc}- u_i\left(\frac{tv^2}{4\alpha}\right)^{1/2}\operatorname{erfc}\frac{1}{2C}(-\epsilon+u_iC^2) - \left(\frac{4\alpha}{\pi tv^2}\right)^{1/2}\Bigl[\frac{C}{(\pi)^{1/2}}\exp\left(-\frac{1}{4C^2}(-\epsilon+u_iC^2)^2\right) - \\
&\frac{1}{2}(-\epsilon+u_iC^2)\operatorname{erfc}\frac{1}{2C}(-\epsilon+u_iC^2)+Q(\epsilon,t)\Bigr]\Bigr\}+\frac{\alpha}{2vB}\sum_{i=1}^{6}\frac{u_i(u_i+1)(1-u_i^2)}{\prod\limits_{j=1}^{6}(u_j-u_i)}\exp\left[-(1-u_i^2)\frac{tv^2}{4\alpha}-\frac{\epsilon}{2}(1+u_i)\right]\times \\
&\operatorname{erfc}\left(\frac{\alpha}{4tv^2}\right)^{1/2}\left(\epsilon-\frac{u_itv^2}{\alpha}\right) - 2\sum_{i=1}^{6} \frac{u_i(u_i+1)}{\prod\limits_{j=1}^{6}(u_j-u_i)}\exp\left[-(1-u_i)\frac{tv^2}{4\alpha}-\frac{\epsilon}{2}(1+u_i)\right]\operatorname{erfc}-\left(\frac{v^2t}{4\alpha}\right)^{1/2}\operatorname{erfc}\left(\frac{\alpha}{4tv^2}\right)^{1/2}\times \\
&\left(\epsilon-\frac{u_itv^2}{\alpha}\right)-2\sum_{i=1}^{6} u_i^2\frac{(u_i+1)}{\prod\limits_{j=1}^{6}(u_j-u_i)}\exp\left[-(1-u_i^2)\frac{v^2}{4\alpha}(t+t_0)-\frac{\epsilon}{2}(u_i+1)\right]\Bigl\{\operatorname{erfc}\left(\frac{u_iC}{2}\right)\operatorname{erfc}\left(\frac{\alpha}{4tv^2}\right)^{1/2}\times \\
&\qquad\left(\epsilon-\frac{u_itv^2}{\alpha}\right)-\left(\frac{4\alpha}{\pi tv^2}\right)\Bigl[\frac{C}{(\pi)^{1/2}}e^{-(u_i^2C^2/4)}-\frac{1}{2}u_iC^2\operatorname{erfc}\left(\frac{u_iC}{2}\right)+R(\epsilon,t)\Bigr]\Bigr\}_{i\neq j} \qquad (21)
\end{aligned}
$$

The expression for the function $P(\epsilon, t)$, $Q(\epsilon, t)$, and $R(\epsilon, t)$ is rather complex; hence for clarity they appear explicitly in Appendix D.

For simplified burning behavior, it is sufficient to study the resulting temperature perturbation at the surface of the propellant. Under this requirement, the perturbed temperature $\delta T'(o, t)$ is

$$
\begin{aligned}
\delta T'(0, t) = &\frac{-2nc_p|\delta p'/p'|}{c_s[n+2+(E_f/RT_f)][T_w-T_i]}\times \\
&\sum_{i=1}^{6}\frac{u_i(u_i+1)(1-u_i^2)}{\prod\limits_{j=1}^{6}(u_j-u_i)}\exp\left[\frac{tv^2}{4\alpha}(u_i^2-1)\right]\times \\
&\operatorname{erfc}- u_i\left(\frac{tv^2}{4\alpha}\right)^{1/2} \qquad (22)
\end{aligned}
$$

since $P(0, t) = Q(0, t) = R(0, t)$. Before proceeding on to an analysis of Eq. (22), it should be noted that, as $t_0 \to \infty$, the equation for the temperature is similar to that given in Ref. 5.

Fortunately, the time-dependent terms in Eq. (22) may be approximated by the following asymptotic relationships developed in Ref. 8:

$$\operatorname{erfc} z \simeq 1/z(\pi)^{1/2}e^{-z^2} \qquad \text{for} \qquad |\arg z| < \pi/2$$

and

$$\operatorname{erfc} - z \simeq e^{-z^2}/(\pi)^{1/2} \qquad \text{for} \qquad |\arg\text{-}z| < \pi/2 \qquad (23)$$

Hence

$$\lim_{t\gg} e^{tv^2u_i^2/4\alpha}\operatorname{erfc}-u_i\left(\frac{tv^2}{4\alpha}\right)^{1/2} \to \frac{1}{-u_i(\pi tv^2/4\alpha)^{1/2}} \text{ for } Re(-u_i) > 0$$

and

$$\lim_{t\gg} e^{tv^2u_i^2/4\alpha}\operatorname{erfc}\text{-}u_i\left(\frac{tv^2}{4\alpha}\right)^{1/2} \to \frac{e^{tv^2u_i^2/4\alpha}}{(\pi)^{1/2}} \text{ for } Re(-u_i) < 0 \qquad (24)$$

Theoretical Results

From the asymptotic expressions, Eqs. (23) and (24), a component of the perturbed surface temperature will increase in a pure exponential manner when the roots u_1 and u_2 are real and when $u_1 > 0$; $u_2 < 0$; $u_1^2 - 1 > 0$ (or $u_1 < 0$; $u_2 > 0$; $u_2^2 - 1 > 0$). For computational purposes, the explicit expression for Ψ is introduced in Eq. (18); consequently the roots $u_{1,2}$ can be written as

$$u_{1,2} = A[E(t_0) - b] \mp (\{1 + A[E(t_0) - b]\}^2 - 4AE(t_0))^{1/2}$$

where

$$E(t_0) = \frac{1}{C(\pi)^{1/2}}e^{-C^2/4} + \frac{1}{2}\left[1 + \operatorname{erfc}\left(\frac{C}{2}\right)\right]$$

and

$$b = \frac{2c_pT_f}{c_s[n+2+(E_f/RT_f)][T_w-T_i]} \qquad (25)$$

If the term under the square root sign is equated to zero, then the restrictions on $u_{1,2}$ are satisfied as well as the stipulation $u_1^2 - 1 > 0$. Hence

$$1 + A\{E(t_0) - b\} = 2[AE(t_0)]^{1/2} \text{ with } A \geq 1/4E(t_0) \qquad (26)$$

Figures 1 and 2 are the graphs for Eq. (26) for different values of $C = [v^2t_0/\alpha]^{1/2}$. For comparison, Fig. 1 shows Denison and Baum's results for steady state. The computations indicate that when $C \geq 6$, for all practical purposes, the propellant is at its steady-state condition. From the preceding development, it is apparent that the region above the higher boundary curves of $C \geq 6$, $C = 0.5$, and $C = 1.0$, constitutes a domain of pure exponential increase with time for the perturbed temperature.

It is also feasible that a component of the temperature perturbation will oscillate with two options: firstly, a bounded exponential increase of the oscillation's amplitude with time; secondly, a bounded exponential decrease of the oscillation's amplitude with time. In the former case, it is necessary that

$$Re(-u_1) < 0 \qquad Re(-u_2) < 0$$

and

$$Re(u_1^2 - 1) < 0 \qquad Re(u_2^2 - 1) < 0$$

The boundary curves for this behavior are obtained from the last restriction; hence the expression for the curve is

$$A = (E(t_0) + b)/[E(t_0) - b]^2 \qquad A < 1/4E(t_0) \qquad (27)$$

The lower bounding curve for the family curves $C \geq 6$, $C =$

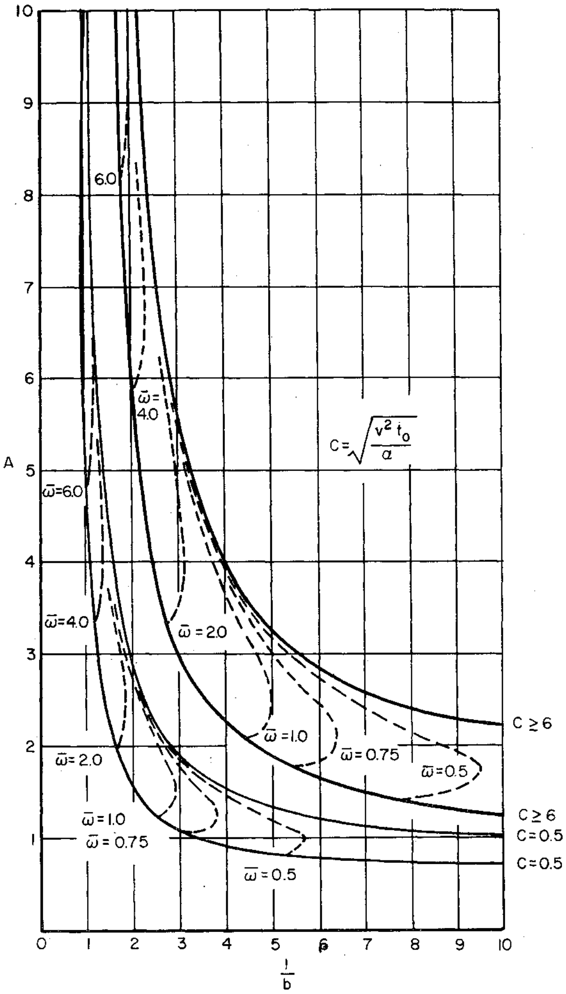

Fig. 1 Stability curves for solid propellant behavior when $C \geq 6$ and $C = 0.5$.

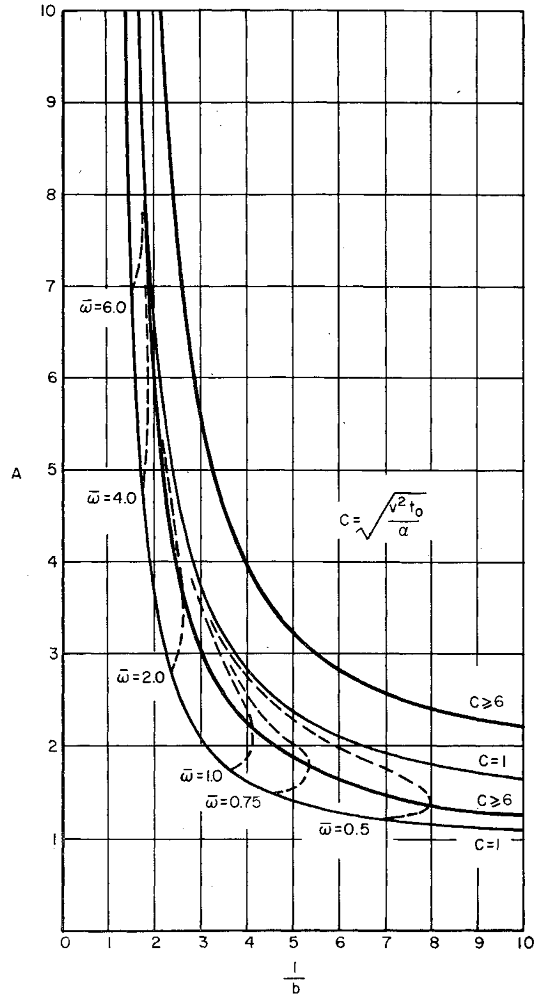

Fig. 2 Stability curves for solid propellant behavior when $C = 1.0$.

1.0, and $C = 0.5$, as shown in Figs. 1 and 2, distinguish the different oscillatory patterns. For points below the curves, the temperature oscillations are damped, and for points between the extremes of any particular curve family, there is a component of the temperature that oscillates with increasing amplitude. In other words, the propellant burns in a stable manner for points in the region of damped oscillations.

For the oscillatory, exponential increase of the temperature amplitude, the associated frequencies are obtained from the relationships

$$Re\{u_{1,2}{}^2 - 1\} \leq 0$$

or

$$\bar{\omega} = \omega\alpha/v^2 = A/2[E(t_0) - b](4AE(t_0) - \{1 + A[E(t_0) - b]\}^2)^{1/2} \quad (28)$$

The curves of constant frequency are indicated in Figs. 1 and 2 as dashed curves.

Since the expression for the surface temperature perturbation, Eq. (22), depends upon the values of the roots u_3 through u_6, the temperature is really composed of a pure oscillatory component in addition to the unstable or stable mode. The former component is referred to as the steady-oscillatory mode, see Ref. 5, and its explicit formulation is obtained from Eq. (22). Thus for the steady-oscillatory mode of the perturbed temperature

$$\delta T'(x, t) = nb\left|\frac{\delta p'}{p'}\right| \frac{r_1}{r_2} \sin\left\{\omega t - \phi_1 + \phi_2 - \frac{\pi}{2}\right\} \quad (29)$$

where

$$r_1{}^2 = \left\{1 + \left[\frac{1 + (1 + 16\bar{\omega}^2)^{1/2}}{2}\right]^{1/2}\right\}^2 + \tfrac{1}{2}\{-1 + (1 + 16\bar{\omega}^2)^{1/2}\}$$

$$r_2{}^2 = \left\{A[E(t_0) + b] - A[E(t_0) - b] \times \left[\frac{1 + (1 + 16\bar{\omega}^2)^{1/2}}{2}\right]^{1/2}\right\}^2 + \left\{2\bar{\omega} - A\,[E(t_0) - b]\left[\frac{-1 + (1 + 16\bar{\omega}^2)^{1/2}}{2}\right]^{1/2}\right\}^2 \quad (30)$$

$$\phi_1 = \arctan \times \left\{\frac{2\bar{\omega} - A[E(t_0) - b][-1 + (1 + 16\bar{\omega}^2)^{1/2}/2]^{1/2}}{A[E(t_0) + b] - A[E(t_0) - b][1 + (1 + 16\bar{\omega}^2)^{1/2}/2]^{1/2}}\right\}$$

$$\phi_2 = \arctan\left\{\frac{\left[-1 + \left(\frac{1 + 16\bar{\omega}^2}{2}\right)^{1/2}\right]^{1/2}}{1 + \left[1 + \frac{(1 + 16\bar{\omega}^2)^{1/2}}{2}\right]^{1/2}}\right\}$$

Since Eq. (29) is a function of the oscillatory frequency $\bar{\omega}$, there exists a frequency for which the temperature exhibits a maximum. The maximizing values are obtained by equating the derivative with respect to $\bar{\omega}$ of the amplitude in Eq. (29) to zero. Figures 3 and 4 illustrate this effect. The dashed lines represent the frequencies that produce the largest value of $|\delta T'/nb|\delta p'/p'||$ and the solid curves give the associated numerical values for the amplitude. It should be noted that the curve $|\delta T'/nb|\delta p'/p'|| = \infty$ is simply the lower bounding curve shown in Figs. 1 and 2.

Discussion of Results and Conclusions

As anticipated, the expression for the perturbed temperature is far more complex than the steady-state solution of Ref. 5. However, at the surface, the temperature expressions are remarkably similar. In the main, there are two major differences in the two expressions. The present analysis has developed an asymptotic solution as against the exact, steady-state solution. In addition, the former method involves roots u_i, which are dependent upon the initiation time t_0. The steady-state solution does not incorporate any effect of t_0. The transient solution reveals an important feature of solid propellant behavior which is not revealed by the steady-state analysis. The graphs shown in Figs. 1 and 2 demonstrate that the propellant behavior is dependent upon the initiation time of the pressure oscillation. In other words, a propellant with $A = 2$, $b = 0.25$ will behave unstably if a pulse is produced at $C = 0.5$; however, it will burn stably when the propellant is subjected to the same pulse at a later time $C \geq 6$. As a further illustration of this important phenomenon, the stability curves in Fig. 2 overlap each other. Hence, the point $A = 2$, $b = 0.25$ can be associated with either a stable behavior or an oscillatory exponential increasing mode. Again, the dimensionless time parameter C determines the mode of behavior. For the steady-state condition, the graphical results for $C \geq 6$ apply, and they are coincident with Denison and Baum's results.

It should be noted that, for a propellant whose physical properties are such that it lies in the region above the upper branch of the curves $C \geq 6$, this propellant will always burn unstably, i.e., continuous exponential increase. The initiation time does not influence any point in this region. Furthermore, for the same time t_0, propellants exhibiting high burning velocities v and low thermal diffusivities α are more likely to burn stably since the exponential unstable area diminishes for larger values of C. On the other hand, the region of damped stable and oscillatory exponential burning will increase for higher values of C. Figures 1 and 2 also indicate that for the latter mode of behavior the frequencies become larger for increasing values of C.

In Figs. 3 and 4, the resonant amplitudes and frequencies for the temperature component of pure oscillation are plotted for two values of the dimensionless time parameter C. For a cosine pressure perturbation, the effect of varying C is more complex. If the propellant properties are in the region above the infinite amplitude curve, then the amplitude $\delta T'/nb|\delta p'/p'|$ and frequency $\bar{\omega}$ increase for larger values of C. However, for a point located below the infinite amplitude curve, the reverse is true: larger values of C produce a decrease in frequency and oscillation amplitude.

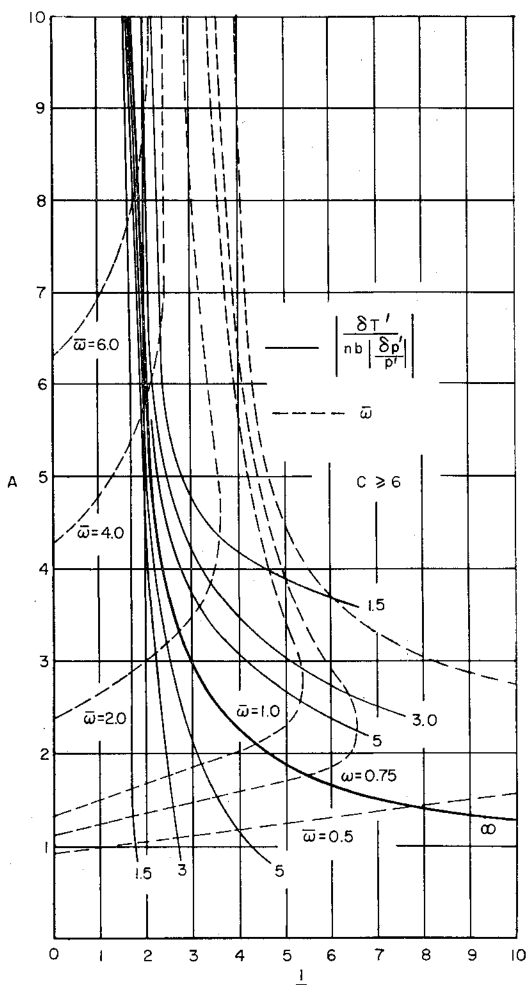

Fig. 3 Resonant amplitude and frequency curves for solid propellant behavior when $C \geq 6.0$.

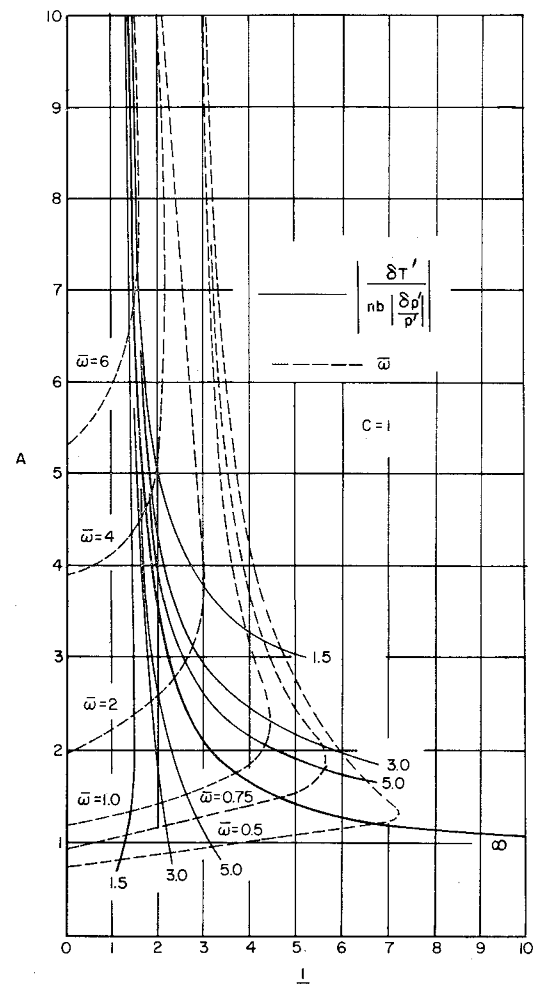

Fig. 4 Resonant amplitude and frequency curves for solid propellant behavior when $C = 1.0$.

Table 1 Initiation times t_0

v, in./sec	ρ, lb/in.^{3}	k^a, Btu/hr-ft-°F	$c_s{}^a$ Btu/lb-°F	α, ft^2/hr	t_0, sec
0.142	0.059	0.25	0.35	7.02×10^{-3}	2.73×10^{-1}
0.291	0.063	0.25	0.35	6.58×10^{-3}	1.1×10^{-1}
0.192	0.06	0.25	0.35	6.9×10^{-3}	2.68×10^{-1}

[a] Assumed.

Before proceeding further, it should be recognized that the present development incorporates certain tacit assumptions. These are, in the main, the same as those made in Ref. 5. Particularly, the relaxation time of the gas phase must be much less than that of the solid phase, which in turn must be less than the initiation time of the perturbation t_0. Furthermore, the initiation time must be greater than the time for pressurization of the chamber. From the preceding conditions, it follows that there is a limitation upon the realizable oscillatory perturbed temperature frequencies, i.e., the reciprocal of the frequency ω must be greater than the relaxation time of the gas phase. This limitation suggests that the present results are most applicable for low-frequency instability. In addition, the analysis considers that the pressure perturbation is prescribed; consequently, the instability modes are predicated upon the temperature or mass efflux trends rather than upon the pressure. Obviously, serious fluctuations in these quantities will induce variations in the gas pressure; however, the exact manner by which this occurs is not considered in the present investigation. It is also rather interesting to observe that the instability curves shown in Figs. 1 and 2 are independent of the particular formulation for the pressure perturbation.

In summary, the present investigation represents a first step in clarifying the transient behavior of the solid propellant. Actually, any temperature perturbation produces a time-dependent variation in the solid, since the associated energy of the perturbation is absorbed by the propellant's mass. The interaction between the temperature perturbation and the dominant solid propellant temperature field is complex; consequently an analysis of Eq. (9) is difficult. Instead, a simplified treatment is developed so as to examine the time dependency in greater detail. The results of the theoretical treatment demonstrates that, for the assumed model, the instability modes can be influenced by the different temperature fields. In other words, the instability frequencies vary with the energy absorbed by the solid propellant; subsequently, the frequencies are functions of time. As future work, a rigorous treatment of Eq. (9) will be attempted. A series of temperature perturbation pulses will be introduced, and the temperature field will be adjusted in accordance with the results of the present analysis. Thus, the current investigation serves as a base from which the complete behavior of $\delta T'$ may be evolved.

Lastly, the theory suggests that in some propellants the restriction of a steady-state temperature field is too severe. Certainly, the damping effect of the perturbation by the propellant is eliminated by the assumption of steady state. Also, it is not apparent that the propellant is in this condition at perturbation, unless it is postulated beforehand. However, the analysis does provide a method for evaluation of this effect. From the criterion of $C = 6$, a maximum theoretical value of the initiation time t_0 may be computed. Hence, time values less than t_0 indicate a transient condition. Furthermore, it must be remembered that t_0 represents the time period for which Eq. (10) applies; consequently, any time required for ignition, flame spreading, etc., must be excluded. In order to demonstrate the feasibility of a transient field at pressure perturbation initiation, Table 1 has been constructed based upon the data shown in Ref. 11, and the values are compared with the published test pressure trace. Because the time lapse prior to pressure growth is not indicated, an estimate of this quantity is required. A search of the literature indicates that, as a rule of thumb, this value is approximately the same as the time required for the pressure growth. Subsequently, by comparison of the computed and test data, it appears that the first and last propellants possessed a transient field, even with the gross assumption for the time lapse.

There are, of course, propellants that may have large values for the ratio v^2/α; under these circumstances, the steady-state condition is achieved in a very short time. Hence, the results of the transient situation do not apply to propellants in this category.

Appendix A

The denominator in Eq. (16) is

$$\frac{v}{\alpha}e^{st_0}\left\{1-\frac{p}{q}\right\}\operatorname{erfc}\frac{C}{2}\left(1+\frac{4s\alpha}{v^2}\right)^{1/2} - \left\{\frac{q}{2B}+\frac{v^3}{\alpha^2 s}\left[1+\frac{q}{2}\right]\left[1+\operatorname{erf}\left(\frac{C}{2}\right)\right]+\frac{2v\psi}{\alpha}\right\} \quad \text{(A1a)}$$

Upon substitution of $u^2 = 1 + 4s\alpha/v^2$, (A1a) becomes

$$\frac{v}{\alpha}\exp\left[\frac{u^2C^2}{4}-\frac{C^2}{4}\right]\left\{\frac{2u}{u+1}\right\}\operatorname{erfc}\left(\frac{uC}{2}\right)+\frac{u+1}{2B}+ \frac{2v}{\alpha(u+1)}\left[1+\operatorname{erf}\left(\frac{C}{2}\right)\right]-\frac{2v\psi}{\alpha} \quad \text{(A1b)}$$

As shown in Ref. 9 for large values of z, the error function may be approximated by the asymptotic series

$$\operatorname{erfc} z \approx \frac{e^{-z^2}}{2(\pi)^{1/2}}\left\{1-\frac{1}{2z^2}+\frac{1.3}{(2z^2)^2}\cdots\right\} \quad |\arg z| < \frac{3\pi}{4} \quad \text{(A1c)}$$

Neglecting all terms beyond the first, it is apparent that

$$\operatorname{erfc}\left(u\frac{C}{2}\right) \approx \frac{1}{(\pi)^{1/2}}\frac{e^{-(uC/2)^2}}{uC/2} \quad \text{(A1d)}$$

Substitution of (A1d) into (A1b) simplifies the denominator to

$$\frac{2B(u+1)}{u^2+2u-\dfrac{4Bv\psi u}{\alpha}+1+\dfrac{8Be^{-C^2/4}}{(\alpha\pi t_0)^{1/2}}+\dfrac{4vB}{\alpha}\left[1+\operatorname{erfc}\left(\dfrac{C}{2}\right)\right]-\dfrac{4Bv\psi}{\alpha}} \quad \text{(A1e)}$$

The denominator in (A1e) is a simple quadratic expression; hence

$$u_{1,2} = A\psi - 1 \mp \left\{(A\psi)^2 - \frac{4A}{C(\pi)^{1/2}}e^{-C^2/4} - 2A\left[1+\operatorname{erf}\left(\frac{C}{2}\right)\right]\right\}^{1/2} \quad \text{(A1f)}$$

Appendix B

The inverse of Eq. (17) may be obtained by successive application of the shifting theorem and LaPlace transform relationships Hence

$$L_\eta^{-1}\, ug(u^2) = \frac{2vB}{\alpha}\sum_{i=1}^{6} \frac{u_i(u_i^2-1)}{\prod_{j=1}^{6}(u_j-u_i)} \exp\left[-\frac{C^2}{4}(1-u_i^2) - \frac{\epsilon}{2}(1-u_i) + u_i\eta\right]\left\{-\operatorname{erfc}\frac{1}{2C}[2\eta+\epsilon+u_iC^2] + \operatorname{erfc}\frac{1}{2C}[\epsilon+u_iC^2]\right\} + \frac{2vB}{\alpha}\sum_{i=1}^{6}\frac{u_i(u_i+1)^2}{\prod_{j=1}^{6}(u_j-u_i)}\exp\left[-\frac{C^2}{4}(1-u_i^2) - \frac{\epsilon}{2}(1+u_i) + u_i\eta\right]\left\{-\operatorname{erfc}\frac{1}{2C}[2\eta-\epsilon+u_iC^2] + \operatorname{erfc}\frac{1}{2C}[u_iC^2-\epsilon]\right\} + \frac{4vB}{\alpha}\sum_{i=1}^{6}\frac{u_i(u_i+1)}{\prod_{j=1}^{6}(u_j-u_i)} e^{-\epsilon+u_i\eta}\operatorname{erfc}\frac{1}{2C}[\epsilon - C^2] + e^{-\epsilon/2}\, U\left(\eta-\frac{\epsilon}{2}\right)\times \left\{\sum_{i=1}^{6}\frac{u_i(u_i+1)(1-u_i^2)}{\prod_{j=1}^{6}(u_j-u_i)} e^{u_i[\eta-(\epsilon/2)]} - \frac{4vB}{\alpha}\sum_{i=1}^{6}\frac{u_i(u_i+1)}{\prod_{j=1}^{6}(u_j-u_i)} e^{u_i[\eta-(\epsilon/2)]}\operatorname{erfc}-(C) - \frac{4vB}{\alpha}\sum_{i=1}^{6}\frac{u_i^2(u_i+1)}{\prod_{j=1}^{6}(u_j-u_i)}\times \exp\left[u_i\left(\eta-\frac{\epsilon}{2}\right) - \frac{C^2}{4}(1-u_i)^2\right]\left[-\operatorname{erfc}\frac{1}{2C}(2\eta+u_iC^2-\epsilon) + \operatorname{erfc}\left(\frac{u_iC}{2}\right)\right]\right\} \qquad i\neq j \tag{B1}$$

Appendix C

A large portion of the terms in Eq. (B1) can be integrated directly. Hence

$$\left(\frac{4\alpha}{\pi tv^2}\right)^{1/2}\int_0^\infty \exp\left[-\frac{\eta^2}{tv^2} + u_i\eta\right]d\eta = e^{u_i^2tv^2/4\alpha}\operatorname{erfc}-\left(\frac{u_i^2tv^2}{4\alpha}\right)^{1/2} \tag{C1a}$$

and

$$\left(\frac{4\alpha}{\pi tv^2}\right)^{1/2}\int_0^\infty \exp\left[-\frac{\eta^2\alpha}{tv^2} + u_i\eta\right]U\left(\eta-\frac{\epsilon}{2}\right)d\eta = e^{u_i^2tv^2/4\alpha}\operatorname{erfc}\left(\frac{\alpha}{4tv^2}\right)^{1/2}\left[\epsilon - \frac{u_itv^2}{\alpha}\right] \tag{C1b}$$

The integration of the term

$$\left(\frac{4\alpha}{\pi tv^2}\right)^{1/2}\int_0^\infty \exp\left[-\frac{\eta^2\alpha}{tv^2} + u_i\eta\right]\times \operatorname{erfc}\frac{1}{2C}[2\eta+\epsilon+u_iC^2]d\eta \tag{C1c}$$

is tractable if (C1c) is rewritten as

$$\left(\frac{4\alpha}{\pi tv^2}\right)^{1/2} e^{-u_i^2tv^2/4\alpha}\int_0^\infty \exp\left\{-\frac{\alpha}{tv^2}\left[\eta - \frac{1}{2}\frac{u_itv^2}{\alpha}\right]^2\right\}\times \operatorname{erfc}\frac{1}{2C}[2\eta+\epsilon+u_iC^2]d\eta \tag{C1d}$$

and noting that

$$\exp\left\{-\frac{\alpha}{tv^2}\left[\eta-\frac{1}{2}\frac{u_itv^2}{\alpha}\right]^2\right\} = 1 + \sum_{n=1}^{\infty}(-1)^n\frac{\left(\frac{\alpha}{tv^2}\right)^n\left(\eta-\frac{1}{2}\frac{u_itv^2}{\alpha}\right)^{2n}}{n!} \tag{C1e}$$

The integration of the first term in straightforward; thus

$$\int_0^\infty \operatorname{erfc}\frac{1}{2C}[2\eta+\epsilon+u_iC^2]d\eta = -\frac{C}{(\pi)^{1/2}}\exp\left\{-\frac{1}{4C^2}[2\eta+\epsilon+u_iC^2]^2\right\} + \frac{1}{2}[2\eta+\epsilon+u_iC^2]\operatorname{erfc}\frac{1}{2C}[2\eta+\epsilon+u_iC^2]\Big|_0^\infty \tag{C1f}$$

The remaining terms in the summation are prepared for integration by an initial integration by parts

$$\int_0^\infty\left(\eta-\frac{1}{2}\frac{u_itv^2}{\alpha}\right)^{2n}\operatorname{erfc}\frac{1}{2C}[2\eta+\epsilon+u_iC^2]d\eta = \frac{[n-\frac{1}{2}u_itv^2/\alpha]^{2n+1}}{2n+1}\operatorname{erfc}\frac{1}{2C}[2\eta+\epsilon+u_iC^2]\Big|_0^\infty + \frac{1}{C(\pi)^{1/2}}\left(\frac{2}{2n+1}\right)\int_0^\infty\left(\eta-\frac{1}{2}\frac{u_itv^2}{\alpha}\right)^{2n+1}\times \exp\left[-\frac{1}{4C^2}(2\eta+\epsilon+u_iC^2)^2d\eta\right] \tag{C1g}$$

From Ref. 10

$$\int(z-\alpha)^ne^{-h^2(x-\beta)^2}dz = \sum_{\nu=0}^{n}\binom{n}{\nu}\times \frac{(\beta-\alpha)^{n-\nu}}{h^{\nu+1}}\int y^\nu e^{y^2}dy \qquad y = h(x-\beta) \tag{C1h}$$

and

$$\int z^ne^{-z^2}dz = -e^{-z^2}\sum_{\nu=0}^{r-1}\frac{(n-1;\ -2;\ \nu)}{2^{\nu+1}}z^{n-2\nu-1} + \frac{(1-s)(1;2;r)}{2^{r+1}}(\pi)^{1/2}\operatorname{erf} z \qquad n = 2r - s \tag{C1i}$$

$$s = 0 \text{ or } 1$$

Applying Eqs. (C1h) and (C1i) to Eq. (C1g)

$$\int_0^\infty\left(\eta-\frac{1}{2}\frac{u_itv^2}{\alpha}\right)^{2n+1}\exp\left[-\frac{1}{4C^2}(2\eta+\epsilon+u_iC^2)^2\right]d\eta = \sum_{\nu=0}^{2n+1}\binom{2n+1}{\nu}\frac{(-\frac{1}{2}u_itv^2/\alpha - [(\epsilon+u_iC^2)/2])^{2n+1-\nu}}{(\alpha/v^2t_0)^{(\nu+1)/2}}\times \left\{\frac{(\pi)^{1/2}(1-s)(1;2;r)}{2^{r+1}}\operatorname{erfc}\frac{1}{2C}(\epsilon+u_iC^2) + \exp\left[-\frac{1}{4C^2}(\epsilon+u_iC^2)^2\right]\sum_{a=0}^{r-1}\frac{(\nu-1;-2;a)}{2^{a+1}}\times \left[\frac{1}{2C}(\epsilon+u_iC^2)\right]^{\nu-2a-1}\right\} \tag{C1j}$$

Equation (21) is obtained by combining the preceding formulations in addition to reapplying the integration methods to the other integral expressions.

Appendix D

For use in Eq. (21), the functions $P(\epsilon, t)$, $Q(\epsilon, t)$, and $R(\epsilon, t)$ are

$$P(\epsilon, t) = \sum_{n=1}^{\infty} \left(\left[\frac{(-1)^n}{n!}\left(\frac{\alpha}{tv^2}\right)^n\right]\right)\left(\frac{-[-\frac{1}{2}(u_i tv^2/\alpha)]^{2n+1}}{2n+1} \operatorname{erfc}\frac{1}{2C}(\epsilon + u_i C^2) + \frac{1}{C(\pi)^{1/2}}\left(\frac{2}{2n+1}\right) \times \right.$$
$$\sum_{\nu=0}^{2n+1}\left\{\left(\frac{\binom{2n+1}{\nu}}{(C^2)^{(\nu+1)/2}}\right)\left[-\frac{1}{2}\frac{u_i tv^2}{\alpha} - \left(\frac{\epsilon + u_i C^2}{2}\right)\right]^{2n+1-\nu}\right\}\left\{\frac{(\pi)^{1/2}(1-s)(1;2;r)}{2^{r+1}} \operatorname{erfc}\frac{1}{2C}(\epsilon + u_i C^2) - \right.$$
$$\left.\left.\exp\left[-\frac{1}{4C^2}(\epsilon + u_i C^2)^2\right]\sum_{a=1}^{r-1}\frac{(\nu-1;2;a)}{2^{a+1}}\left(\frac{1}{2C}[\epsilon - u_i C^2]\right)^{\nu-2a-1}\right\}\right) \quad \text{(D1a)}$$

$$Q(\epsilon, t) = \sum_{n=1}^{\infty} \left(\left[\frac{(-1)^n}{n!}\left(\frac{\alpha}{tv^2}\right)^n\right]\right)\left(\frac{-[-\frac{1}{2}u_i(tv^2/\alpha]^{2n+1}}{2n+1} \operatorname{erfc}\frac{1}{2C}(-\epsilon + u_i C^2) + \frac{1}{C(\pi)^{1/2}}\left(\frac{2}{2n+1}\right) \times \right.$$
$$\sum_{\nu=0}^{2n+1}\left\{\left(\frac{\binom{2n+1}{\nu}}{(C^2)^{(\nu+1)/2}}\right)\left[-\frac{1}{2}\frac{u_i tv^2}{\alpha} - \left(\frac{-\epsilon + u_i C^2}{2}\right)\right]^{2n+1-\nu}\right\}\left\{\frac{(\pi)^{1/2}(1-s)(1;s;r)}{2^{r+1}} \operatorname{erfc}\frac{1}{2C}(-\epsilon + u_i C^2) - \right.$$
$$\left.\left.\exp\left[-\frac{1}{4C^2}(-\epsilon + u_i C^2)^2\right]\sum_{a=1}^{r-1}\frac{(\nu-1;-2;a)}{2^{a+1}}\left[\frac{1}{2C}(-\epsilon + u_i C^2)\right]^{\nu-2a-1}\right\}\right) \quad \text{(D1b)}$$

$$R(\epsilon, t) = \sum_{n=1}^{\infty} \left(\left[\frac{(-1)^n}{n!}\left(\frac{\alpha}{tv^2}\right)^n\right]\right)\left(\frac{-\left[\frac{\epsilon}{2} - \frac{1}{2}\left(\frac{u_i tv^2}{\alpha}\right)\right]^{2n+1}}{2n+1} \operatorname{erfc}\left(\frac{u_i C}{2}\right) + \frac{1}{C(\pi)^{1/2}}\left(\frac{2}{2n+1}\right) \times \right.$$
$$\sum_{\nu=0}^{2n+1}\left\{\left(\frac{\binom{2n+1}{\nu}}{(C^2)^{(\nu+1)/2}}\right)\left[-\frac{1}{2}\frac{u_i tv^2}{\alpha} - \left(\frac{-\epsilon + u_i C^2}{2}\right)\right]^{2n+1-\nu}\right\}\left\{\frac{(\pi)^{1/2}(1-s)(1;2;r)}{2^{r+1}} \operatorname{erfc}\left(\frac{u_i C}{2}\right) + \right.$$
$$\left.\left.e^{-(u_i C/2)^2}\sum_{a=0}^{r-1}\frac{(\nu-1;-2;a)}{2^{a+1}}\left[\frac{u_i C}{2}\right]^{\nu-2a-1}\right\}\right) \quad \text{(D1c)}$$

for $\nu = 2r - s$ and $s = 0$ or 1.

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